How Do Mathematics Teachers Utilize Their Specialized Content Knowledge In Teaching The Concept of Derivative?

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***Abstract*—The Specialised Content Knowledge (SCK) is an important sub-domain under Mathematical Knowledge for Teaching (MKT) framework that describes the mathematical content knowledge that is unique to the profession of teaching and not necessarily required by other professions in which mathematics is used. Research on SCK in the context of secondary education is still underdeveloped while SCK of in-service teachers have also received limited attention. The current study investigates the SCK of Sri Lankan in-service secondary mathematics teachers in teaching an important Calculus concept in G.C.E. (Advanced Level) combined mathematics curriculum: the concept of Derivative. Lessons of ten mathematics teachers from seven schools were observed. The data comprises of field notes and video recordings of lessons and these video recordings were transcribed verbatim for analysis. Transcripts of lessons were analysed through the lens of Ball's MKT framework while placing more emphasis on SCK sub-domain. Analysis of data identified four different components that describe SCK in teaching the concept of derivative, namely: SCK-PK (Building new knowledge on prior knowledge), SCK-MR (Multiple Representations), SCK-MJ (Mathematical Justifications), and SCK-ML (Mathematical Language). Teachers exhibited each SCK component in varying degrees of proficiency. Majority of teachers elicited their content knowledge under SCK-MR and SCK-ML components while gaps were noticed in SCK-MJ and SCK-PK components. This study sheds light on the current status of G.C.E. (Advanced Level) combined mathematics teachers with respect to SCK and provides insights for planning teacher education programs and workshops.**

Keywords—Mathematical knowledge for teaching, specialized content knowledge, The concept of derivative, secondary teaching

I. Introduction

Teacher knowledge is an essential ingredient in teaching mathematics and it is empirically evident that teacher knowledge has a direct impact on quality of instruction and student achievements [1, 4]. Due to this reason, understanding the knowledge demands in teaching mathematics has gained an attention among educators during the past few decades. However, due to its multi-dimensional nature, assessing teacher knowledge is often a challenging task. In the past, teacher knowledge has been assessed directly, through proxy variables such as degrees obtained, number of courses completed or standardized test scores of teachers [1, 4]. Most academics realized such measures as problematic [1] as they do not sufficiently reflect teacher's subject matter content knowledge *for* teaching mathematics. However, a new line of thinking came into light with Shulman's [2] argument that "knowledge for teaching mathematics" is different from "knowledge for doing mathematics". Shulman [2] identified that the subject matter knowledge and pedagogical knowledge that a teacher must possess are not mutually exclusive but is a blended special set of skills and knowledge in order to teach, which he coined as Pedagogical Content Knowledge (PCK). His seminal work on PCK sparked an interest in many researchers to investigate content knowledge *for* teaching and various models emerged to define and categorize the knowledge demands to carry out the work of teaching mathematics. Such categorization of teacher knowledge permits educators and policy makers to understand the current standing of mathematics teachers and to provide better insights into planning teacher education and professional development programs.

A. MKT Model

Shulman's [2] seminal work on PCK was brought into and adopted by a Ball and her colleagues [3], a group of researchers at Michigan University. They conducted a longitudinal research to ascertain the type of content knowledge that matters for teaching and revealed that the typologies suggested by Shulman [2] as vital to leverage the content knowledge of a teacher as the technical knowledge in defining teaching as a profession. Based on the working definition for Mathematical Knowledge for Teaching (MKT) as "mathematical knowledge needed to carry out the work of teaching mathematics" [3, p. 395], they collected and analysed the work carried out by teachers (a job analysis or a bottom-up practice based approach) throughout a year by way of video and audio tapes of lessons, students work, homework, lesson plan, etc. Their final model, the MKT framework is a practice-based theory that encapsulates the mathematical knowledge needed to perform the recurrent tasks of teaching mathematics. Although MKT theory was first developed in the U.S. context for primary teachers, it was later adapted to assess the MKT of elementary and secondary teachers and has been widely used in many other continents. The current study will opt for Ball et al.'s MKT model as an analytical framework due to its' comprehensive structure and empirical validity in assessing teacher knowledge.

According to MKT framework, teacher knowledge demands in teaching mathematics can be categorized into two main domains (see Fig. 1): Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK). Each of these domains consists of three sub-domains. Sub-domains under SMK are Common Content Knowledge (CCK): the mathematical knowledge commonly used by settings other than teaching, Specialised Content Knowledge (SCK): the mathematical knowledge that is unique to the profession of teaching, and Horizon Content Knowledge (HCK): a peripheral vision of mathematics or the knowledge of how the currently taught content is connected to larger mathematical ideas and structures. Sub-domains that belong to PCK are, Knowledge of Content and Teaching (KCT): the knowledge in planning and designing the instruction, Knowledge of Content and Students (KCS): the knowledge that intertwines the knowledge about mathematics with the knowledge about students such as anticipate students' reactions to a particular task, student's common mathematical errors and misconceptions and finally, Knowledge of Content and Curriculum (KCC): knowledge on instructional materials, curriculum and programs.

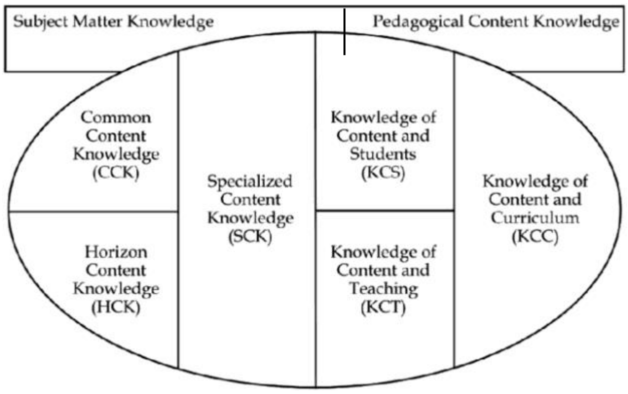

Fig. 1. Domains and Sub-domains of MKT Model [3]

Our study aimed at investigating the SCK in the context of senior secondary teaching in Sri Lanka. There were several underlying reasons opting for SCK over other domains in MKT framework. Scholars [3, 4] have placed a special interest on the study of SCK due to its relative importance in teaching and have acknowledged the need of future work due to its contribution for content preparation in teacher development. In addition, SCK encompasses "purely" mathematical knowledge which is unique to the profession of teaching. Hence, studies on SCK allow researchers to ascertain the topic specific content knowledge of teachers and provide a direction for educators and policy makers to better orient teacher education, development and training initiatives for prospective teachers.

B. *Specialised Content Knowledge (SCK)*

The Specialised Content Knowledge (SCK) is an important sub-domain (out of six sub-domains – see Fig. 1) under MKT framework which describes the content knowledge that is unique to the profession of teaching and "not typically needed for any purpose other than teaching" [3, p. 400]. Ball, Thames and Phelps [3] listed tasks of teaching that are associated with SCK which includes (a) linking representations to underlying ideas, (b) finding examples to make a mathematical point, (c) giving mathematical explanations, (d) presenting mathematical ideas, (e) asking productive mathematics questions, (f) inspecting equivalencies, (g) choosing or developing usable definitions, etc. It also includes the way teacher conducts "error analysis" - recognizing and rectifying student errors promptly, appraising and analysing unconventional solution methods that students present, sizing up the source of error, justifying generalizations, etc. As mentioned in [3, p. 397] "this is the type of work that teachers must do rapidly, often on the fly, because in a classroom, students cannot wait as a teacher puzzles over the mathematics". However, the description of Ball, Thames and Phelps [3] for SCK does not restrict to have such knowledge for mathematicians, but emphasized that having such knowledge is *not essential* for mathematicians to do their jobs while it is *compulsory* for teachers in teaching mathematics. According to [5], an accountant and a doctor do not necessarily require a mathematical reasoning for using common denominator when adding two fractions (although they have all rights to know it), but it is indeed a natural work in a classroom teaching. One of the easiest ways to recognize SCK over other knowledge domains (especially from PCK) is that SCK is primarily based on the knowledge of the content and doesn't necessarily expect having additional knowledge about students or teaching [3].

C. *Mathematics in Sri Lankan Advanced Level Curriculum*

Advanced Level (A/L) combined mathematics syllabus in Sri Lanka constitutes of two components: Pure and Applied Mathematics. Major topics covered under pure mathematics are Algebra, Calculus, Coordinate Geometry and Trigonometry out of which Calculus has gained far more attention in the curriculum due to several reasons. Firstly, calculus is a fundamental branch in mathematics and is a pre-requisite for many advanced topics in mathematics. Secondly, calculus section carries a significant weightage in the General Certificate of Education A/L examination, contributing to more than 25% of total marks. Thirdly, calculus is a completely new area for collegiate students as they learn it for the first time at A/L. All these facts signify the importance of calculus in A/L curriculum. Calculus comprises of four sections: Limits, Derivatives, Application of Derivatives and Integration. However, its inherent abstract nature, mathematical jargons and symbols makes it difficult for beginners to understand. Therefore, a special attention is required to ensure that these concepts are taught well at A/L classrooms. The current study aimed at investigating the teacher knowledge based on the introductory lesson of the derivative and to ascertain teachers' level of SCK elicited during instruction. This study will answer two research questions given below.

1. What are the components of SCK in teaching the concept of derivative for secondary students?
2. In the identified components of SCK, what are the strengths and weaknesses demonstrated in the instruction?

II. METHODOLOGY

A. *Participants*

This study followed a qualitative approach and the data was drawn from a two year funded research on exploring the Mathematical Knowledge for Teaching (MKT) of A/L mathematics teachers in teaching limits and derivatives. First, a list of potential participants was prepared based on referrals from university lecturers, teachers and educators who conduct professional development programs for teachers. Neither teaching experience nor any other special characteristic was considered in selecting participants. Next, these participants were contacted to check their interest on taking part in the study. At the time we initiated this study (in year 2021), majority had already completed limit and derivative lessons. Therefore, selection process had to be iterated until the stipulated number was met. Finally, 16 advanced level mathematics teachers working in 11 different government and private schools across three districts in Sri Lanka gave their consent to participate in this two year research project. Ten out of 16 teachers participated in teaching the "introduction to the derivative", a topic which is usually taught during the 3rd term of year 12. Usual time allocation to teach the introductory lesson of derivatives is six periods (6 x 40 minutes).

B. Data Collection

Data were primarily collected through lesson observations and field notes. Collection of data did not disturb the usual flow of how curriculum is lined-up in a usual academic year. Instead, classes were visited as and when the particular lesson was being conducted. Through continuous follow-ups with teachers, a schedule was prepared and lesson observations were conducted accordingly. Two teachers conducted their lessons in English medium while other teachers did their lessons in Sinhala medium. These lessons were video recorded for the purpose of transcribing and indebt analysis. Through manual transcribing (verbatim) process, all video recordings were converted into text file format which contain teacher-student conversations (with line numbers), time stamp, utterances and gestural information as well as the diagrams, tables and other data captured through the audio. In order to maintain participants' privacy, all the teachers were given pseudonyms (from A to P).

C. Data Analysis

Lesson transcripts were analyzed qualitatively and Ball, Thames and Phelps's [3] MKT framework was primarily employed as the guiding theory in analysing these transcripts. To answer the first research question of "what are the components of SCK in teaching the derivative for secondary students", we first scanned the data through the lens of MKT framework to explore the tasks of teaching belonging to the six different MKT domains. An in-depth understanding of each subdomain in MKT framework was required in order to differentiate and distinguish SCK from other sub-domains.

After reading the transcripts several times to understand the general behaviour of the data, we identified meaningful chunks of words (A chunk consists of statements, explanations, a conversation between teacher and students, questions asked, etc.) in such a way that such chunk of words is small enough to explore components of SCK and large enough to explore at least one or two SCK components. At this stage, notes have been taken based on identified patterns and similarities and reflected back to the research question to identify initial codes. For instance, if a unit of analysis (or chunk of words) describes teacher *work* on graphical representation of the derivative, such unit was coded as "graphical representation". However, if that chunk of words also illuminates the teacher knowledge on choosing examples wisely to make an important mathematical point, that chunk was coded again as "example to make mathematical point". Hence, a single unit of analysis could have more than one code. However, if this word chunk describes teacher knowledge that belongs to any other sub-domain other than SCK, such chunks directly coded based on the name of that sub-domain(s) (e.g. HCK, CCK, KCS, KCC or KCT). For example, if a teacher discusses about structure of the curriculum, such word chunk was given a code as KCC as the teacher knowledge on curriculum structure belongs to KCC domain. This method facilitated in filtering out SCK easily from the rest of the sub-domains. However, a deep understanding of the MKT framework was needed to undertake this process as there were instances where overlapping occurs (e.g. SCK with CCK or SCK with KCT).

While using MKT framework as a lens to explore data, we also utilized an inductive-deductive mixed approach to identify *open* and *priori codes* to explore patterns of data read through the transcripts.

III. RESULTS

After a series of refinements, we were able to identify various tasks of teaching that reflect MKT of a teacher. However, to be in line with the scope and the research questions of the study, we will be reporting the routine but unique tasks of teaching that require SCK to execute. Hereafter, we will refer these routine tasks of teaching as *components* of SCK. Tab. 1 outlines the components of SCK that emerged during the qualitative analysis.

TABLE I. COMPONENTS OF SCK IN TEACHING THE DERIVATIVE

Component	Meaning
SCK-PK:Building new knowledge on prior knowledge	The content knowledge of a teacher in building new knowledge on prior knowledge.
SCK-MR: Multiple representations	The content knowledge of a teacher in using symbolic, graphical and verbal representations.
SCK-MJ: Mathematical Justification	The content knowledge of a teacher in providing valid mathematical justifications for actions undertaken during instruction.
SCK-ML: Mathematical Language	The content knowledge of a teacher in using topic specific lexical and natural language.

SCK-PK represents the knowledge of the teacher in activating and linking prior knowledge with new knowledge. Ball, Thames and Phelps [3] listed "connecting a topic being taught to topics from prior to future years" as a task that demands SCK and this aligns with the SCK-PK of the current study. SCK-MR is commonly emphasized in general educational research, MKT research [3, 6] and educational standards such as NCTM [5] as a key component in teaching. Ball, Thames and Phelps [3] identified, *linking representations to underlying ideas and to other representations* as a teaching task requiring SCK and [6] highlighted *representation* as one of the central components of SCK. The component, SCK-MJ centers on how teachers utilize their SCK to justify mathematical procedures, explain conceptual meaning and reasoning, etc. Justification has been vastly emphasized as a significant aspect in teaching mathematics [3, 6]. Moreover, teachers' expertise in explaining and justifying mathematical ideas (e.g. the rationale behind inverting and multiplying to divide by fractions) reflects their level of SCK [3]. SCK-ML was also emerged through data which aligns with the SCK component noted by Ball, Thames and Phelps [3] who once mentioned "teachers, however, must be able to talk explicitly about how mathematical language is used (e.g. how the mathematical meaning of edge is different from the everyday reference to the edge of a table)" [3, p.400].

Despite contextual differences, the alignment between SCK components in our study (as listed in Table 1) and Ball, Thames and Phelps's [3] empirical results are noteworthy. These components strengthen the applicability of MKT model (developed based on primary teachers in United States) to the secondary teaching context in Sri Lanka with a few refinements. In the next section, we discuss the strengths and gaps revealed under each SCK component in order to answer the second research question of this study.

A. SCK-PK: Building New Knowledge on Prior Knowledge

As illustrated in Fig. 2, learning the concept of derivative demands a significant portion of prior knowledge on functions, algebra, trigonometry, etc. which are logically interwoven with the concept of derivative.

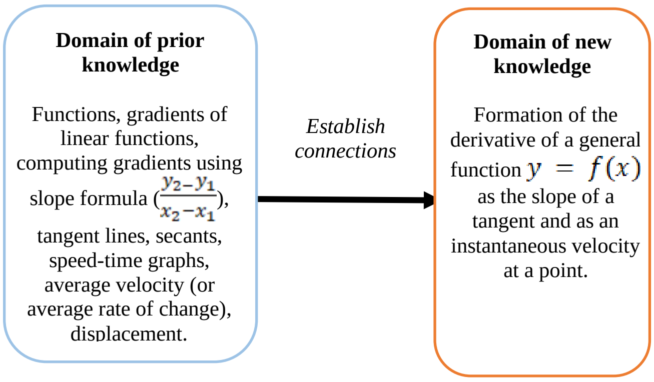

Fig. 2. Linking prior knowledge to new knowledge

However, data provides evidence that although teachers recalled certain mathematical ideas as an when it was required, majority of them were not particularly keen on activating prior knowledge at the very beginning of the lesson. Instead, most of them initiated the lesson by directly deriving the formal definition of derivative. Only two teachers (teachers A and I) activated topic specific prior knowledge, however their focus on establishing explicit links between the existing and new knowledge was not adequate. As a result of that, important learning opportunities were missed and learning was deprived of establishing meaningful connections across the mathematical concepts. Information becomes meaningful when the brain links new information with the prior knowledge and teachers as moderators could expedite this process by recalling prior topics. If teachers fail to do so, it may encourage rote learning as new learning occurs independently with no reference to prior knowledge. This section of the analysis clearly points the significant gaps in teachers' SCK in recalling the essential prior knowledge to learn the concept of derivative, and therefore SCK-PK deserves much attention in secondary context.

B. SCK-MR: Multiple Representations

When a mathematical concept is represented in a variety of forms, it provides learners an opportunity to observe and better understand a concept through multiple facets which could promote deep learning of abstract mathematical concepts. Derivative is a concept that can be introduced in variety of forms. Similar to the illustration of teacher E (see Fig. 3), all other participating teachers took efforts to integrate verbal, graphical and symbolical representations interchangeably when presenting the definition of the derivative.

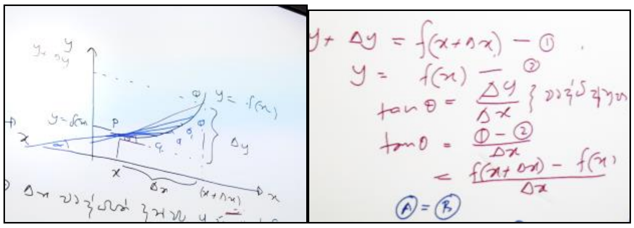

Fig. 3. Graphical and Symbolic representations–Work of teacher E

Graphical method is considered as a powerful form of representation that enables students to visualize abstract mathematical concepts in a concrete manner. A prominent observation was that the majority of the teachers represented the concept of derivative through a series of well-elaborated diagrams and most of them moved flexibly within and in between other modes of representations. Fig. 4 depicts the diagrams used by four teachers to explain the concept of derivative.

However, we also noticed few incomplete diagrams (in the cases of teacher A, N, O & I) which lacked basic features in representing the derivative. For instance, as evident in Fig. 4, teachers A and I did not draw series of secant lines, but used hand movements and verbally explained how secant line (e.g *PQ*) gradually approaches "the tangent line at *P*". In addition to that, as in Fig. 5, teacher N's graphical demonstration did not contain the basic features to represent the derivative and she also failed to illustrate the same verbally. Instead, she followed a direct approach to introduce $\frac{\Delta y}{\Delta x}$ with no reference to the gradient of the secant line or tangent line. Such representations tend to add less visibility to the concept and tend to leave doubts or induce misconceptions among learners. According to the analysis, diagrams used by some teachers (Teachers C, E and J) elicited more elaborative power when compared to the diagrams used by other teachers (Teachers A, N, O & I). The gaps in SCK-MR of teachers A, I and N were largely evident through their graphical illustration of the derivative.

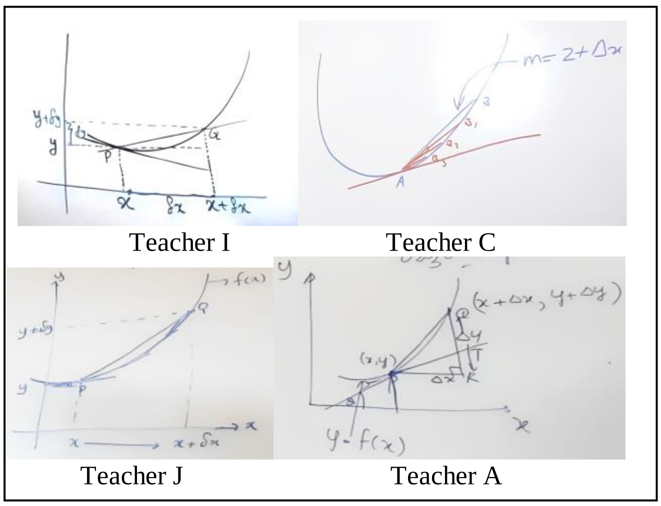

Fig. 4. Diagrams used by four teachers to introduce the derivative

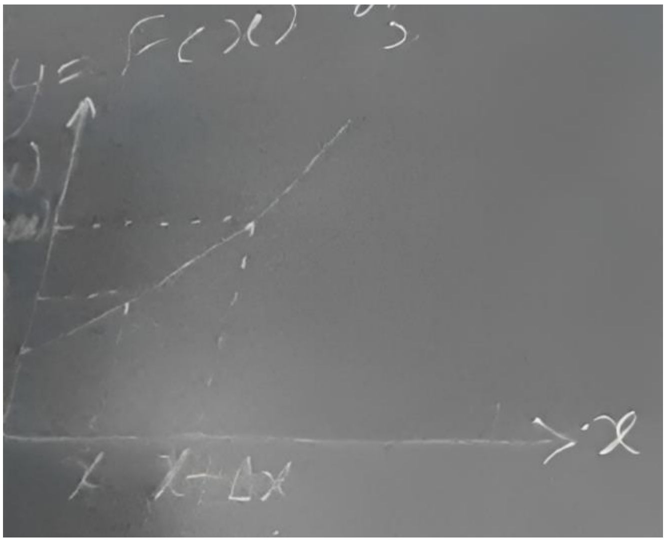

Fig. 5. Diagram used by teacher N to introduce the derivative

Symbolic representation also plays a pivotal role in presenting the definition of the derivative $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$, the definition which is used to manipulate derivatives through first principles. All ten participants were well aware of the symbolic representation of the derivative and took efforts to integrate it simultaneously to the graphical representation. After converting the coordinates into symbols, teachers then explained verbally, the method of computing the gradient of the secant line *PQ* by drawing a triangle. Few teachers clearly represented the gradient of the secant either through the ratio $\frac{rise}{run} = \frac{\Delta y}{\Delta x}$ or by $\tan\theta = \frac{\Delta y}{\Delta x}$. Following this, they switched to symbolic representation to develop two equations in order to derive an expression for $\frac{\Delta y}{\Delta x}$.

Their next step was to explain the role of limit. Some teachers switched to graphical form again to demonstrate how Δx approaches zero and teachers E, C and J drew series of secant lines (three or four lines) until *Q* approached *P* and they also used hand movements to demonstrate how secant line gradually approaches *P* until it is about to overlap with tangent at *P*. Other teachers (A, G, H, I, O) graphically and (or) verbally demonstrated how Δx approaches 0 by referring to the Δx distance marked on the *x*-axis. After incorporating limits, they finally reached the final symbolic version of the definition of derivative $\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ provided the limit exits. While verbally explaining the concept displayed through the graphs, teachers simultaneously translated verbally and graphically presented ideas into symbols. It was also evident that majority of them moved flexibly between symbolic, verbal and graphical representation and tried to establish links between representations (see Fig. 6).

However, it was noticed that none of the participants, even the ones who were privileged to use SMART boards, took efforts in integrating computer generated diagrams to explain the main idea behind the derivative, instead used traditional static graphs to demonstrate the concept.

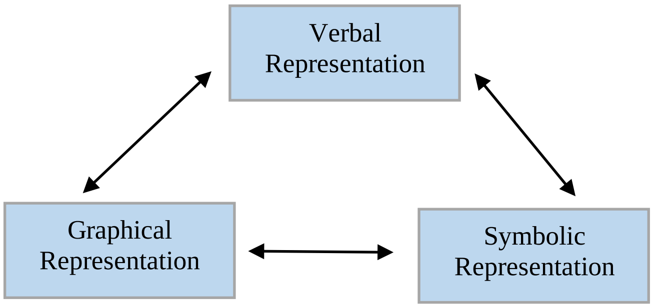

Fig. 6. Establishing links between representations

C. *SCK-MJ: Mathematical Justification*

SCK-MJ represents the knowledge of the teacher in providing valid mathematical justifications for actions undertaken during instruction. For the purpose of this study, we used the definition of justification outlined in [7, p.448] as "an argument that demonstrates (or refutes) the truth of a claim that uses accepted statements and mathematical forms of reasoning". Deriving the formal definition of derivative encompasses many steps that build on several mathematical concepts such as average rate of change, gradient of a secant line, the concept of limit, etc. We will outline the level of SCK of teachers in justifying five main actions/steps found in instruction namely: (1) drawing a tangent line (at *P*), (2) choosing another point (say *Q*) on the curve $y= f(x)$, (3) drawing a secant line *PQ*, (4) writing an expression for difference quotient and (5) applying limits to difference quotient.

A derivative lesson usually begins with considering the problem of defining the gradient of a curve $y = f(x)$ at a point *P* on it (or the gradient of tangent line to the curve at *P*). Students need to understand that, unlike for a straight line, gradient of a curve is not constant and therefore the gradient of a curve at a point cannot be simply computed using the general gradient formula. However, it was noticed that, all teachers (except teacher I) failed to do this comparison. In order to establish valid connections between topics, it is of paramount importance to emphasize the difference between the gradient of a linear function and a curved function. It is also important to convey that average rate of change and instantaneous rate of change are identical for linear functions and are not necessarily identical for non-linear functions.

The next common step noticed was drawing a tangent line at *P*. All teachers drew a tangent line to the curve without mentioning that it is the very line whose gradient needs to be defined. There is no way to draw the tangent without **knowing** its gradient.

Further to that, learners need to understand that the gradient of the tangent at *P* is still unknown (since they only know one point on the tangent line) which can be approximated by the gradient of the secant line *PQ* (or average rate of change). However, such reasoning was lacking in many lessons. Teachers directly plotted the second point (say *Q*) on the curve by moving Δx along the *x*-axis from *P* and then drew a line segment by connecting *Q* with the fixed-point *P*. However, except teacher C, all other teachers chose the second point *Q* without giving a valid reasoning for such action. To understand their actions, the excerpt of teacher E is given below.

Teacher E, Line 3:

> *So now I mark two points on this curve... one is P... other one is called Q…then if this point P is x…then correspondingly what will happen to the point here* [teacher locates a point on *y* axis] *...y is equal to f*(*x*)*...do you understand? This point Q is not very far from P…very close...when x is given an increment called delta x... that means when we give a very small change, it will go to point Q...is it clear?...then what are the coordinates of point Q here?.*.[Contd…]

As in the above explanation, teacher E directly marked the two points *P* and *Q* on the curve $y = f(x)$, identified the coordinates of *Q* as $(x+\Delta x, y+\Delta y)$, followed by writing the difference quotient. All teachers failed to mention that the gradient of the secant line *PQ* doesn't accurately represent the gradient of the tangent at *P*, but it is just an approximation. None of them justified how this approximation gets improved as *Q* gets closer to *P* along the curve. "Approximation" is an important terminology that needs to be introduced at this point of the lesson. However, the justification for choosing *Q* closer enough to *P* was barely discussed. Teachers selected a nearby point *Q* to *P* (from Δx distance along the *x*-axis) and also stressed that Δx needs to be a very small distance, but failed to mention the reason for taking a small distance instead of a large distance.

Teacher H, Line 10:

> *Now I said that this delta x gap is a very small gap…it is very close to* x_0 *..now I make this delta x… smaller and smaller...and then I see what happens to this to the ratio of delta x over delta y when delta x becomes smaller and smaller.. What happens when it is reduced?* [Contd…]

As in the above excerpt, teacher H just mentioned "delta *x* gap is a very small gap", but he failed to mention reason for opting for a small distance delta *x*. Finally, in applying limits, most teachers mentioned that the secant line *PQ* overlaps with tangent line at *P* as delta *x* approaches zero. However, none of them explained that the gradient of the secant line *PQ* becomes a good approximation for the gradient of the tangent line at *P* as *Q* becomes close enough to *P*. More technically, as *Q* reaches *P*, the average rate of change gradually becomes a better approximation for the instantaneous rate of change. The only way to achieve this is applying the limit which can be mathematically represented as $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$. Apparently, many teachers missed this opportunity to point out an important application of limit.

Overall, it was revealed that majority of the teachers' mathematical work were algorithmic and those procedures were barely justified. These results are striking and indicate that many teachers had the primary intension to derive the formal definition of derivative by following a common algorithm, thereby lost the opportunity to provide conceptual understanding of the derivative. However, if the teachers elaborate "whys of what they're doing" [7, p.448], it scaffolds learning and develops life-long skills like mathematical language development, communication, critical thinking and independence. Further to the above analysis, it is also believed that teachers strictly adhering to the main guidelines outlines in the teacher guide and following the exact steps in it. This may prevent them from thinking and elaborating on the mathematical procedures. However, teachers need to be aware that teaching guides only outline the main content in the lesson and that they need to draw from their SCK in supporting ideas, justifications and reasoning.

D. *SCK-ML: Mathematical Language*

The final SCK component related to teaching the concept of derivative is SCK-ML which discusses the knowledge of teachers in utilizing mathematical language in instruction, focusing on vocabulary, symbols, and natural language. However, we will not particularly discuss the symbolic usage in this section as it was already being outlined under SCK-MR.

The finding revealed both positive aspects and deficiencies under SCK-ML. On the positive side, it was noticed that majority of the teachers were fairly conversant with the mathematical symbols and vocabulary and they placed a considerable attention to their language in teaching derivatives. As listed below, majority of the teachers adequately explained the new terminology pertaining to the lesson derivative.

1) *Average Rate of Change*

One out of ten teachers (teacher I) mentioned about average rate of change which is one of the most important phrases to be recalled and utilized when finding the gradient of the secant. This outcome is consistent with the fact that most teachers did not recall prior knowledge (SCK-PK) on gradient formulae for straight line nor explicitly mentioned the gradient of the secant as the average rate of change.

2) *Delta x*

Majority of the teachers adequately and repeatedly explained the meaning of Δx and Δy. Most of them used the word phrases like "a very small change", "a small distance", etc. to elaborate Δx and the evidence for such discussions are given below.

Teacher E, Line 3:

> "*When x is given delta x increment…that means* ***a very small change****…*"

Teacher G, Line 6:

> "*Generally when x changes…accordingly y changes. ..isn't that so?* ***A small change in x*** *is known as an increment in x and that denotes by delta x. ok? Therefore, delta y is the increment of y corresponding to delta x increment in x".*

3) *Difference Quotient or Increment Ratio*

Only a very few participant teachers (E, G, J and N) displayed their mathematical language fluency in introducing technical names for $\frac{\Delta y}{\Delta x}$ as *difference quotient, increment ratio or ratio between increments* while all others just read the symbolic notation as delta *y* over delta *x*.

Teacher E, Line 9:

"*So delta y means the increment in y… divided by delta x which is the increase in x... is that clear? So what do we say for this? Difference quotient... Difference quotient…*"

4) *Differential Coefficient*

In Calculus, $\frac{dy}{dx}$ is given denoted by several names. Therefore, it is important to explicitly introduce the words and their meanings, otherwise the complexity of such words would hinder understanding the overall concept of derivative. Many teachers used "first derivative" or simply "the derivative" to denote $\frac{dy}{dx}$ while few teachers used the words, "instantaneous rate of change", "derived function", "differential coefficient" to introduce the same and applied such words interchangeably in their discussions to make sure that students got familiar with the new terminology and their meanings. Excerpts of such discussions are as follows.

Teacher I, Line 10:

"*However we can take it as limit delta x goes to zero delta y over delta x. this limit zero delta y over delta x goes to zero called the* ***instantaneous rate of change***."

Teacher E, Line 17:

"*So what is defined by the* ***first differentiation*** *that is dx by dy? gradient of a curve*"

Teacher G, Line 11:

"*Now if this limit exists… then that is defined as the* ***differential coefficient*** *or derivative of function f at x…*" [Contd..] … *"or you can call it as* ***first derivative****"*

Teacher G, L12:

"*f dash x or dfx by dx is called the derived function of f*(*x*)*. Ok? we derived it from f*(*x*)...[Contd...]"

Based on the above findings, it was also evident that many teachers elicited their content knowledge on the topic specific vocabulary pertaining to the concept of derivative, critically evaluated the meaning, discussed its' usage and repeatedly used such vocabulary during instruction. Such work of teachers reflected their effort in augmenting students' mathematical lexicon.

Few flaws were also noticed from some teachers' due to slip of the tongue and use of ambiguous language with demonstrative pronouns like *this, that and these*. This hinders the precision of ideas conveyed and students can be easily misled. In such cases, teachers missed opportunities in creating mathematically rich discussions. Mathematics is a universal language, so everyone who uses it needs to adhere to standard notations, syntax and vocabulary to articulate ideas precisely. However, when the language is vague and imprecise, the overall meaning gets distorted. Most notable outcome was that teachers who exhibited stronger knowledge in other SCK components (i.e. SCK-PK, SCK-MR and SCK- MJ) also exhibited a high level of proficiency in SCK-ML. Specifically, teachers C, D, E and H exhibited their proficiency in the use of mathematical language throughout the entirety of their lessons, with minor lapses. They ensured the use of accurate phrases and vocabulary that are comprehensible to learners. The results shed light into the importance of the mathematical language in delivering complex mathematical ideas. In order to ensure error-free discussions, teachers need to stick to domain specific vocabulary, use them precisely and consistently, practice with articulating ideas and need to refrain from oversimplifying or overgeneralizing mathematical concepts through colloquial terms.

Overall, this research focused on the *unique work of teaching* to understand the current standing of A/L mathematics teachers in terms of SCK. Their unique mathematical work employed during the derivative instruction reflected several strengths of SCK that support student learning as well as deficiencies that hinder learning. Most notable finding was that a relationship was apparent in between components of SCK and if a teacher is competent in one component, he or she demonstrated proficiency in many other SCK components while the converse was also true.

While acknowledging the fact that A/L teachers deal with a tough classroom schedule and handle immense pressure for the completion of a lengthy syllabus, the pedagogical and content knowledge gaps needs to be addressed immediately for the betterment of both teachers and learners. Owing to the exam oriented teaching-learning setup, teachers tend to prefer procedural aspects than facilitating a conceptual understanding on abstract mathematical concepts. Although not statistically proven, results also demonstrated that years of experience may not always determine teacher knowledge as teachers with decades of teaching experience exhibited significant deficiencies in communicating ideas, justification and conceptual understanding. Nevertheless, professional qualification tends to predict the teacher knowledge since, despite few lapses, teachers with MSc in Mathematics Education demonstrated their proficiency in terms of many SCK components when compared to others who do not possess professional qualification in teaching. Teachers as educators are expected to strengthen the understanding of the mathematical work of teaching (or mathematics *for* teaching) to develop students' broader mathematical landscape. Educators, teachers and policy makers need to clearly distinguish the knowledge requirement of teachers to teach mathematics. It is understood that completion of undergraduate advanced mathematics modules would not suffice, instead, they need to have a deep understanding on the unique mathematical work associated to the profession of teaching.

However, to get better insights this study could have employed more participants. Owing to certain constrains imposed by schools during covid-19 pandemic, we were not able to reach the expected target of 16 participants for the derivative instruction. Further to that, few selected participants were infected hence their lesson were not observed. Another notable, but inherent limitation of this type of study is that observed lessons might have been specifically prepared for recording purposes. Consequently, the observed lesson may not reflect the true nature of their teaching. This was evident from the feedback received from some teachers who admitted that they tend to skip such introductory session on derivatives unless these sections are directly assessed in Advanced Level examination. Hence, future studies could consider employing random classroom observations to explore the true picture of SCK during instruction.

IV. Conclusion

SCK is a specific type of subject matter content knowledge that is essential for mathematics teachers in carrying out the unique mathematical work related to teaching. Through this study, we were able to identify four distinct components of SCK in teaching the concept of the derivative namely: SCK- PK, SCK-MR, SCK- MJ and SCK-ML. Despite contextual differences, the alignment between the proposed SCK components in this study and empirical results in [3] is noteworthy. These results strengthen the applicability of MKT model (originally developed based on primary teachers in United States) to the secondary teaching in Sri Lanka with a few refinements.

It is clear that teachers are the role models in the classroom who shape the teaching-learning process. Hence, it is essential for them to know the content of subject well enough so that they could teach it to others. There focus should not be limited to “how much” content they teach, but “how well” they teach the content. In such nature, teachers cannot afford gaps in their instructions. However, the above qualitative analysis revealed various proficiencies as well as deficiencies in SCK during their practice. To address these identified gaps, it is a timely strategy to launch workshops and seminars that are tailor made to uplift necessary content knowledge required for teaching. Further to that, education policy makers need to take prompt initiatives to design mathematics content courses which are free from typical content under university mathematics courses but to capture the mathematical knowledge demands for future teachers. It was evident that teachers require necessary guidance to overcome their flaws in the subject matter knowledge. Hence, it is vital to foster collaboration among teachers, teacher educators, lecturers and other mathematics experts in knowledge within the community. Also, it was noticed that teachers who had access to such tools through smartboards utilized them solely as conventional whiteboards for writing purposes. Therefore, it is important for schools to take proactive training initiatives for teachers to optimize the use of interactive technology to facilitate teaching-learning process.

This study contributes new knowledge to the existing body of literature that focuses on exploring and developing the content knowledge of mathematics teachers and specifically in the area of SCK. While previous studies have predominantly employed Ball, Thames and Phelps’s [3] MKT framework to gauge the knowledge of primary or elementary level teachers, this study extended the model to senior secondary context by examining SCK from a different perspective. Furthermore, this study extended upon the *routine tasks,* outlined in [3], that demands SCK and proposed a list of SCK components specifically tailored to assess the knowledge in teaching the concept of derivative. The development of such SCK framework serves multiple purposes including acting as a guideline to identify the SCK of teachers in a more objective manner, ensures consistency in evaluating SCK across different contexts or settings and facilitates replication of the same research with new samples of teachers from other districts in Sri Lanka to draw more comprehensive and generalized conclusions. It is worth noting that, while there is an abundance of research focusing on MKT of prospective student (teachers), limited research exists on in-service teachers. This study was able to address this gap by exploring how experienced (in-service) teachers demonstrate their knowledge in an actual classroom setting. Most importantly, to the authors’ knowledge, no study has been conducted in Sri Lanka on exploring MKT of mathematics teachers. Therefore, the results of the current study provide invaluable insights into the current standing our A/L mathematics teachers in terms of their content knowledge and also the level of attention required towards developing their SCK.

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